AN ALGEBRAIC APPROACH TO NECESSITIES

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The metaphysics of modality has traditionally focused on the study of particular notions of necessity, such as metaphysical necessity, nomic necessity, historical necessity, and so on. Recently, however, there has been a shift towards the systematic study of *necessities* as a whole. A big driver of this shift is that some questions about particular notions of necessity are best answered by appealing to the general structure of necessities. Most prominently, Kripke (1980) introduces the notion of metaphysical necessity as "necessity in the highest degree, whatever that means." Systematic theories of necessities have been developed for the explicit purpose of making sense of this idea, by giving precise accounts of what it means for a necessity to be of higher degree or *broader* than another, and what sort of structure necessities form under this relation (Hale, 2013; Williamson, 2016; Bacon, 2018; Roberts, 2025).

There are also many general questions about necessities that are of independent interest, and can be fruitfully studied in abstraction from any particular notion of necessity. For example, what does it mean for a necessity to be a restriction of another (Hale and Leech, 2017; Roberts, 2020; Bacon and Zeng, 2022)? Can all intensional notions be reduced to extensional ones (Lewis, 1986; Bacon, 2018, 2023, Ch. 7)? More generally, clarifying the structure of necessities is interesting in its own right. The concept of a necessity is central to metaphysics—no less central than, say, those of parthood and location—and metaphysics, like any other part of philosophy, should seek as complete an account as possible of the structure of its central concepts.

The theory of necessities developed in Bacon (2023, Ch.7) is an ideal environment to study questions of this sort. Bacon gives a reductive analysis of what a necessity is and what it means for a necessity to be broader than another in purely logical terms, meaning that many questions about the structure of necessities can be answered through logical means alone. The general idea driving Bacon's approach is that necessities are to the world what normal modal operators—the boxes of modal logic—are to language. Bacon makes this intuition precise by internalizing the *proof theory* of normal modal logics. Syntactically, a normal modal logic for an operator M is a set of sentences that (i) contains all tautologies, (ii) contains every instance of the `K`-axiom $M(P \to Q) \to MP \to MQ$, and is closed under the rules of (iii) modus ponens and (iv) necessitation for M: if P is in the logic, so is MP. Mirroring this definition, Bacon defines in the object language the property of being a *normal theory* for a property of propositions M. He then defines a *weak necessity* to be a property of propositions whose every normal theory is true, and a *necessity* proper as something that is necessarily a weak necessity, according to every weak necessity.

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It is natural to consider a different way of implementing the same general idea: instead of internalizing the proof theory of normal modal logics, we can internalize their *semantics*. In this paper, I develop an approach along these lines that internalizes the *algebraic semantics* of normal modal logics. Algebraically, propositional modal languages are interpreted over *modal algebras*: Boolean algebras equipped with an operator—used to interpret the box—that maps the top element to itself and commutes with finite meets. These equational conditions can be straightforwardly internalized in the object language. A necessity, on my account, is simply a property of propositions M that maps tautologies to themselves and commutes with conjunction: $M(P \wedge Q)$ equals $MP \wedge MQ$, for any P and Q.

I show that the algebraic characterization of necessities is equivalent to Bacon's original proof-theoretic one: to be a necessity in the algebraic sense just is to be a necessity in Bacon's sense. I then apply my algebraic characterization of necessities to prove general results about the *algebra of necessity*—the algebraic structure that the family of necessities forms under the entailment relation. Bacon and Zeng (2022) proved that necessities in Bacon's sense form a bounded lattice with respect to entailment, but left open the question of whether the lattice is distributive. My algebraic characterization of necessities affords a simple proof that it is. What is more, I prove that necessities satisfy an infinite distributivity law that gives them the structure of a *Heyting algebra*, the sort of algebraic structure used to interpret the intuitionistic propositional calculus.

With Heyting algebra structure at hand, we can make sense of the operation of taking the *pseudocomplement* of a necessity. If we think of an M-necessary proposition as one that is true at all M-possible worlds, then the pseudocomplement of a necessity M is the property of being true at all M-*impossible* worlds (Humberstone, 1983; Gargov et al., 1987); the pseudocomplement operation is a generalization of this idea that also applies to necessities that cannot be characterized in terms of possible worlds this way.

The pseudocomplement operation does not in general behave classically. I show how to characterize different intuitive senses in which a necessity can be *defective* in terms of which classical principles—like excluded middle and double negation elimination—it fails to satisfy. The property of propositions *having probability 1*, for example, is arguably a necessity in the operative sense, but is defective in at least two ways. First, it is not *principal*: for every proposition that has probability 1, we can find a strictly stronger proposition that also has probability 1; there is no maximally strong proposition with probability 1. Second, it is not *infinitely closed*: it fails to make necessary every logical consequence of the propositions it makes necessary (Bacon and Zeng, 2022; Bacon, 2023, Ch. 7). Many paradigmatic examples of necessity, like metaphysical necessity and nomic necessity, do not seem to be defective in either way. I give a general characterization of pseudocomplements and use it to show that a necessity is principal iff its join with its pseudocomplement is the narrowest necessity (excluded middle), and that a necessity is infinitely closed iff it is equal to its double pseudocomplement (double negation elimination).

1. Preliminaries

We work in a standard simply typed higher-order language. The *types* of our language consist of the basic types e, t of singular terms and sentences respectively, and the functional type $\sigma \to \tau$ for any types σ, τ. We assume there is a countably

$\forall \bar{x} M := \forall x_0 \cdots \forall x_n M$	$\lambda \bar{x}.M := \lambda x_0 \cdots \lambda x_n.M$
$\heartsuit_t := \heartsuit \qquad (\heartsuit \in \{\wedge, \vee, \to, \leftrightarrow\})$	$\heartsuit_{\sigma\to\tau} := \lambda XYz^{\sigma}.Xz\heartsuit_{\tau}Yz$
$\top := \exists^{(e\to t)\to t\to t}\exists^{(e\to t)\to t}$	$\bot := \neg\top$
$\bot_t := \bot$	$\bot_{\sigma\to\tau} := \lambda x^{\sigma}.\bot_{\tau}$
$\top_t := \top$	$\top_{\sigma\to\tau} := \lambda x^{\sigma}.\top_{\tau}$
$=_{\sigma} := \lambda XY.\forall Z^{\sigma\to t}(ZX \leftrightarrow ZY)$	$\subseteq_{\sigma} := \lambda XY^{\sigma\to t}.\forall Z(XZ \to YZ)$
$F[\bar{A}] := FA_0 \wedge \cdots \wedge FA_n$	

FIGURE 1. Abbreviations. Here $\bar{x} = (x_0, \ldots, x_n)$ is any sequence of variables and $\bar{A} = (A_0, \ldots, A_n)$ is any sequence of terms of the same type.

infinite set of variables Var^{σ} for each type σ. A *signature* is a typed family of sets $\Sigma^{\cdot}$ with Σ^{σ} a set of constant symbols for each type σ. The *terms* of our language are defined as those we can recursively build from variables and constants via application and λ-abstraction, in the usual way. We write $A : \sigma$ to indicate that A is a term of type σ. Throughout, we work in an unspecified signature containing the usual Boolean connectives and quantifiers at every type. Some useful abbreviations are collected in Figure 1.

Our background higher-order logic is H, a standard classical higher-order logic in which $\beta\eta$-equivalence implies identity. A full axiomatization can be found in Bacon (2023, Ch. 5). We will work in an extension of H introduced in Bacon and Dorr (2024) called *Classicism* (C), obtained by closing the standard classical higher-order logic H under the schematic rule:

$$\text{if } \vdash P \leftrightarrow Q, \text{ then } \vdash \lambda\bar{x}.P = \lambda\bar{x}.Q, \tag{Equiv}$$

This rule ensures that provably coextensive properties are identical. In particular, it ensures that there is exactly one logical truth, $\top$, as any two logical truths (theorems of H) are provably equivalent in H.

It will be helpful to work with a seemingly more general version of this rule, the *conditional equivalence* rule, which actually turns out to also be a theorem of Classicism.

$$\text{if } \vdash P = \top \to (Q \leftrightarrow Q'), \text{ then } \vdash P = \top \to \lambda\bar{x}.Q = \lambda\bar{x}.Q'. \tag{CEquiv}$$

Here we must also impose the requirement that $\bar{x} \notin \mathsf{FV}(P)$. We thus have a strategy to prove higher-order identities from assumptions, provided we assume that those assumptions enjoy the status of logical truths.

In Classicism, we can give an algebraic definition of *entailment*, both for propositions and for higher-order relations. We let

$$\leq_{\sigma} := \lambda XY.X \wedge_{\sigma} Y = X.$$

2. BACON'S THEORY OF NECESSITIES

Bacon (2023, Ch. 7.2) introduces his logical notion of necessity in Classicism. As noted, the guiding idea is to internalize the proof theory of normal modal logics into the object language. To do that, we first define what it is for a property of propositions to be a *normal theory* for a property of propositions. A normal theory is a worldly analog of the notion of a normal modal logic: a set of sentences containing all tautologies, every instance of the K-axiom, and closed under modus ponens and necessitation. We begin by defining:

$$\mathsf{Nec} := \lambda XY \forall p(Xp \to X(Yp))$$
$$\mathsf{K} := \lambda XY.X(\mathsf{MP}(Y))$$

We then put:

$$\mathsf{NTh} := \lambda X^{t\to t} Y^{t\to t}.\mathsf{Taut}X \wedge \mathsf{MP}X \wedge \mathsf{Nec}XY \wedge \mathsf{K}XY$$

Intuitively, $\mathsf{NTh}XY$ says that X is a normal theory for the operator Y. That is, X applies to the tautologous proposition, is closed under modus ponens and necessitation for Y, and says that Y itself is closed under modus ponens, which mirrors the K-axiom.

Definition 2.1. A *weak necessity* is defined as a property of propositions whose every normal theory is true:

$$\mathsf{WNec} := \lambda X.\forall p \forall Y(\mathsf{NTh}YX \wedge Yp \to p)$$

A *necessity* is a property of propositions that is necessarily a weak necessity, relative to every weak necessity:

$$\mathsf{Nec} := \lambda X.\forall Y(\mathsf{WNec}Y \to Y(\mathsf{WNec}X)).$$

Bacon defines a necessity M to be *broader* than a necessity N if M entails N, *narrower* if M is entailed by N. As is now well known, Classicism proves that there is a *broadest* necessity, $\Box$. It can be defined as the property of having every necessity:

$$\Box := \lambda p.\forall X(\mathsf{Nec}X \to Xp),$$

and shown to be equivalent to the property of being identical to the tautologous proposition,

$$\mathtt{C} \vdash \Box = (\lambda p.p = \top).$$

In other words, the broadest necessity is that which makes only tautologies necessary.

We can use the broadest necessity to characterize entailment as *broadly strict implication*:

$$\mathtt{C} \vdash \leq_\sigma = \lambda XY^\sigma.\Box\forall \bar{z}(X\bar{z} \to Y\bar{z}),$$

where $\sigma = \rho_0 \to \cdots \to \rho_n \to t$ and $\bar{z} = (z_0, \ldots, z_n)$ is a sequence of variables of types $\rho_0, \ldots, \rho_n$ respectively. This makes clear that Classicism comes with a *coarse-grained* metaphysical picture: broadly necessarily equivalent properties and propositions are identified.

3. Algebraic Necessities

It is natural to wonder whether we can get a well behaved notion of necessity by internalizing the *semantics* of normal modal logics, rather than their proof theory. This section develops an approach along these lines that internalizes the *algebraic semantics* of normal modal logics.

The central concept of the algebraic semantics of normal modal logics is that of a *modal algebra* (e.g., Chagrov and Zakharyaschev, 1997, Ch. 7). A modal algebra is simply a Boolean algebra equipped with an operator that maps the top element to itself and commutes with finite meets. This is a purely equational definition that can be internalized in the object language. The idea here is that propositions form a Boolean algebra structure with respect to entailment, where conjunction and negation give meets and complements respectively. An *algebraic necessity*, then, is just a property of propositions that satisfies appropriate generalizations of the defining equations of modal algebras with respect to the underlying Boolean algebra of propositions.

I begin by discussing some complications that arise when attempting to internalize extensional notions (like that of a modal algebra) in the intensional setting of higher-order logic. I then give a precise definition of algebraic necessities, and show that they are precisely the necessities in Bacon's sense. I close by giving two characterization of algebraic necessities of similarly algebraic flavor, in terms of modalized notions of *filter* and *congruence*.

3.1. **From Extensional to Intensional.**

There is an important disanalogy between the concept of a Boolean algebra and the structure that propositions form with respect to entailment. That of a Boolean algebra is an *extensional* concept. A Boolean algebra is a special kind of poset. A poset is a set equipped with a partial order: a reflexive, antisymmetric, and transitive binary relation. Those of reflexivity, antisymmetry, and transitivity are all extensional notions. In a higher-order setting, they are most naturally formalized thus:

$$\begin{aligned}\mathsf{Refl} &:= \lambda R.\forall x.Rxx\\ \mathsf{Antisymm} &:= \lambda R.\forall xy.(Rxy \land Ryx) \to x = y\\ \mathsf{Trans} &:= \lambda R.\forall xyz.(Rxy \land Ryz) \to Rxz.\end{aligned}$$

Propositional entailment satisfies all three conditions. So does the restriction of the *is as large as* relation to planets in our solar system. But there are important differences in how modally robustly these two relations satisfy the three conditions. As it turns out, no two planets in our solar system are exactly the same size. But they could have been (in some sense of 'could'), in which case the *is as large as* relation restricted to planets in our solar system would have failed to be antisymmetric. Moreover, the Earth is in fact larger than Mercury, but things could (in some sense of 'could') have been the other way around. The position that planets occupy in the structure determined by the *is as large as* relation is merely contingent.

Entailment is different in both respects. Entailment is *broadly necessarily* a partial order, in every sense of necessity. And if p entails q, then this holds of broad necessity. The latter does not only hold of propositions that in fact exist. Had there been "new" propositions r and s such that r entails s, this would also hold as a matter of broad necessity.

These modalized order-theoretic features of entailment can be precisely formulated without presupposing Bacon's notion of broad necessity, using higher-order identities instead. The properties of being broadly necessarily reflexive, antisymmetric, and transitive can be equivalently formulated as follows:

$$\begin{aligned}\mathsf{Refl!} &:= \lambda R.(\lambda x.Rxx) = (\lambda x.\top)\\ \mathsf{Antisymm!} &:= \lambda R.(\lambda xy.(Rxy \wedge Ryx)) \leq =\\ \mathsf{Trans!} &:= \lambda R.(\lambda xyz.(Rxy \wedge Ryz)) \leq (\lambda xyz.Rxz)\end{aligned}$$

To bear R to oneself just is to be such that $\top$. To bear R to one another entails to be equal. To be connected by R in two steps entails to be connected by R in one step. We call these notions *intensional* reflexivity, antisymmetry, and transitivity.

The second feature can be captured through the notion of a *persistent relation*. Intuitively, a relation is persistent when, broadly necessarily, if some things have it then they have it as a matter of broad necessity. We can give a mostly equational formulation of this idea as well:

$$\mathsf{Persistent} := \lambda R.(\lambda S.R \leq S) \leq (\lambda S.R \subseteq (\lambda \bar{x}.S\bar{x} = \top))$$

We will call a relation an *intensional partial order* when it is persistent and intensionally reflexive, antisymmetric, and transitive:

$$\mathsf{PO!} := \lambda R.\mathsf{Persistent}R \wedge \mathsf{Refl!}R \wedge \mathsf{Antisymm!}R \wedge \mathsf{Trans!}R.$$

It is straightforward to verify that, in Classicism, entailment is an intensional partial order, at any type.

There is a similar dialectic involving the other components in the definition of a Boolean algebra. A Boolean algebra is a *lattice*, meaning a partial order where any two elements have both a meet and a join. The notions of meet and join are extensional notions. Focusing specifically on meets and joins with respect to entailment, their most natural higher-order regimentation is as follows:

Definition 3.1 (Bounds)**.** A *lower bound* of $Y : \sigma \to t$ *in* Z is something to which Z applies that entails anything that falls under Y, and an *upper bound* of $Y : \sigma \to t$ *in* Z is something to which Z applies that is entailed by anything that falls under Y:

$$\begin{aligned}\mathsf{LB} &:= \lambda xYZ.Zx \wedge \forall w(Yw \to x \leq w),\\ \mathsf{UB} &:= \lambda xYZ.Zx \wedge \forall w(Yw \to w \leq x).\end{aligned}$$

An *extensional meet* Y *in* $Z : \sigma \to t$ is a lower bound of Y in Z that is entailed by every other lower bound of Y in Z, and a *extensional join* of Y *in* $Z : \sigma \to t$ is an upper bound of Y in Z that entails every other upper bound of Y in Z:

$$\begin{aligned}\mathsf{Meet} &:= \lambda xYZ.\mathsf{LB}xYZ \wedge \forall w(\mathsf{LB}wYZ \to w \leq x),\\ \mathsf{Join} &:= \lambda xYZ.\mathsf{UB}xYZ \wedge \forall w(\mathsf{UB}wYZ \to x \leq w).\end{aligned}$$

I adopt the convention of writing $\mathsf{LB}xY$—pronounced *x is a lower bound of Y*—for $\mathsf{LB}xY\top_\sigma$ and similarly for the other cases. A lower (upper) bound of a pair of entities is defined as a lower (upper) bound of the property of being identical to one or the other, and likewise for extensional meets and joins. In these cases we may write $\mathsf{LB}x[y,z]W$ for $\mathsf{LB}x\lambda v.v = y \vee v = zW$, and likewise for the other cases.

It is easy to see that conjunction and disjunction respectively give extensional meets and joins of pairs of entities (at any relational type) with respect to entailment. But again, this is not the whole story. Entities at any relational type have meets and joins with respect to entailment as a matter of broad necessity, and these meets and joins are necessarily given by their conjunctions and disjunctions. To capture this, we can introduce *intensional* notions of bounds, which again we can do in a purely equational way without assuming the notion of broad necessity.[1]

Definition 3.2 (Intensional Bounds)**.** An *intensional lower bound* of $Y : \sigma \to t$ in Z is some X with ZX such that Y entails the property of being entailed by X, and an *intensional upper bound* of $X : \sigma \to t$ in Z is some Y with ZY such Y entails the property of entailing X:

$$\mathsf{LB!} := \lambda xYZ.Zx \wedge Y \leq (\lambda w.x \leq w),$$
$$\mathsf{UB!} := \lambda xYZ.Zx \wedge Y \leq (\lambda w.w \leq x).$$

An *intensional meet* of Y in Z is an intensional lower bound X of Y in Z such that being an intensional lower bound Y in Z entails the property of entailing X, and an *intensional join* of Y in Z is an intensional upper bound X of Y in Z such that being an intensional upper bound of Y in Z entails the property of being entailed by X:

$$\mathsf{Meet!} := \lambda xYZ.\mathsf{LB!}xYZ \wedge (\lambda w.\mathsf{LB!}wYZ) \leq (\lambda w.w \leq x),$$
$$\mathsf{Join!} := \lambda xYZ.\mathsf{UB!}xYZ \wedge (\lambda w.\mathsf{UB!}wYZ) \leq (\lambda w.x \leq w).$$

I use similar notational conventions for intensional bounds as for extensional bounds.

It is helpful to go through some cases where intensional and intensional meets and joins agree, and cases where they come apart. A relation is called *rigid* when, so to speak, it can neither lose nor gain instances across modal space. Formally, a relation is rigid when it is persistent and the converse entailment in the definition of persistence also holds:

$$\mathsf{Persistent} := \lambda R.(\lambda S.R \leq S) = (\lambda S.R \subseteq (\lambda \bar{x}.S\bar{x} = \top))$$

Example 3.3. Suppose we add *Rigid Comprehension* to Classicism, the principle that every property is coextensive with a rigid property. Let X be a persistent property of propositions, and let X' be a rigid property coextensive with X. Then the proposition that every X' is true, i.e., $\forall p.X'p \to p$, is an extensional meet of X, but not an intensional meet. Because X is persistent and the actual Xs are exactly the X's, $\forall p.X'p \to p$ entails every X and is entailed by anything that entails every X. However, were there any "new" Xs, $\forall p.X'p \to p$ might have failed to entail them. So X cannot be entailed by the property of being entailed by $\forall p.X'p \to p$, and thus $\forall p.X'p \to p$ cannot be an intensional meet of X. The intensional meet of X is instead the proposition that every X is true, i.e., $\forall p.Xp \to p$. If, on the other hand, we modify the example so that X is rigid, then $\forall p.Xp \to p$ is both an extensional and an intensional meet of X.

We can introduce the notion of an *intensional lattice* with respect to entailment. It is straightforward to verify that, in Classicism, all the notions of intensional bounds just introduced are persistent relations. So, we can say that an intensional

[1]Intensional joins are introduced by Bacon and Zeng (2022, Sec. 5), who argue that they are the appropriate notions to use in non-extensional contexts drawing from category theory.

lattice consists of a persistent property where, as a matter of broad necessity, any two entities have both an intensional meet and an intensional join. Persistence will guarantee that intensional meets and joins do not change across modal space.

This can be captured equationally as follows:

Definition 3.4 (Intensional Lattice)**.** An *intensional lattice* with respect to entailment is a persistent property X such that the relation two things bear when they are both instances of X is the same as the relation they bear when they have an intensional meet (resp. join) in X:

$$\begin{aligned}\mathsf{Lat!} := \lambda X.\mathsf{Persistent}X \wedge (\lambda yz.X[y,z]) = (\lambda yz.\exists w(\mathsf{LB!}wy, zX \wedge Xw))\\ \wedge (\lambda yz.X[y,z]) = (\lambda yz.\exists w(\mathsf{UB!}wy, zX \wedge Xw)).\end{aligned}$$

We can do similar things with the remaining components of the definition of a Boolean algebra, defining the notions of *being intensionally distributive* (see Section 4.3), *being intensionally bounded* and *having intensional complements.* This ultimately leads to the notion of an *intensional Boolean algebra.* Propositions form an intensional Boolean algebra with respect to entailment.

3.2. **Algebraic Necessities Defined.** When we think about internalizing the defining conditions of modal algebras, we want to make sure our resulting notion of an algebraic necessity is well behaved with respect to the underlying intensional Boolean algebra structure of propositions, not merely the extensional portion thereof. This motivates defining algebraic necessities as follows:

Definition 3.5 (Algebraic Necessity)**.** A property of propositions M is an *algebraic necessity* if it maps the tautologous proposition to itself, and the relation two propositions bear when their conjunction falls under M is the same as the relation they bear when both fall under M:

$$\mathsf{AlgNec}{:} = \lambda M.(M\top = \top) \wedge (\lambda pq.M(p \wedge q)) = (\lambda pq.Mp \wedge Mq)$$

The first conjunct is a straightforward regimentation of the condition on modal algebras that the modal operator maps the top element to itself. An immediate corollary of the broad necessity of identity is that an algebraic necessity necessarily maps the tautologous proposition to itself, in every sense of necessity. It is the second condition on modal algebras that requires a generalization appropriate for intensional contexts. Just like an intensional partial order must broadly necessarily be, say, antisymmetric, an algebraic necessity must not just *in fact* commute with conjunction, i.e.,

$$\forall pq.M(p \wedge q) = Mp \wedge Mq.$$

Rather, this should hold as a matter of broad necessity. The higher-order identity in the second conjunct ensures that this will be so, without presupposing the notion of broad necessity.

Many key results from Bacon's theory of necessities have analogs for algebraic necessities, often with significantly streamlined proofs. Because algebraic necessities are defined purely equationally, we have $\mathsf{C} \vdash \mathsf{AlgNec}M \leftrightarrow (\mathsf{AlgNec}M = \top)$. Using this fact, we can straightforwardly show that $\Box$ is the broadest algebraic necessity, i.e.,

$$\mathsf{C} \vdash \mathsf{AlgNec}\Box;$$
$$\mathsf{C} \vdash \mathsf{AlgNec} \leq (\lambda M.\Box \leq M).$$

Indeed, $\top = \top \leftrightarrow \top$ is a theorem of $\mathtt{H}$, so by (Equiv) we get $\mathtt{C} \vdash \Box\top = \top$. Distribution over conjunction follows similarly from the classical tautology $(p \wedge q) \leftrightarrow (p \wedge q)$ and $\beta\eta$-equivalence. To establish the second claim, note that $\mathtt{C} \vdash \mathsf{AlgNec}M \to p = \top \to Mp$. By (CEquiv) we thus have $\mathtt{C} \vdash \mathsf{AlgNec}M \to (\Box \leq M)$, which yields $\mathtt{C} \vdash \mathsf{AlgNec} \leq (\lambda M.\Box \leq M)$ by (Equiv).

This is no coincidence. Every result from Bacon's theory of necessities transfers to algebraic necessities, because algebraic necessities are the same thing as necessities in Bacon's sense.

Theorem 3.6. *In Classicism, to be an algebraic necessity just is to be a necessity in Bacon's sense:*

$$\mathtt{C} \vdash \mathsf{AlgNec} = \mathsf{Nec}.$$

Proof. ($\Rightarrow$) Assume $\mathsf{AlgNec}M$. We first show that $\Box$ is a normal theory form M. Clearly, $\mathsf{Taut}\Box$ and $\mathsf{MP}\Box$ hold. For $\mathsf{Nec}\Box M$, note that $\forall q(\Box q \to \Box Mq)$ follows from the fact that $\Box$ is the broadest algebraic necessity, plus the easily verified fact that the composition of two algebraic necessities is an algebraic necessity. As for $\mathsf{K}\Box M$, if $Mp \wedge M(p \to q)$, then $M(p \wedge (p \to q))$ by distribution over conjunction. Because $\mathtt{C} \vdash ((p \wedge (p \to q)) \wedge q) = (p \wedge (p \to q))$ we must then have $M((p \wedge (p \to q)) \wedge q)$, and so Mq by distribution over conjunction again. Generalizing, we have shown $\mathtt{C} \vdash \mathsf{AlgNec}M \to \mathsf{MP}M$, whence $\mathtt{C} \vdash \mathsf{AlgNec}M \to \Box\mathsf{MP}M$ by necessitating, distributing and using the fact that algebraic necessities are broadly necessarily algebraic necessities.

Thus, indeed, $\Box$ is a normal theory for M. Now, if p belongs to every normal theory form M, it must also belong to $\Box$. Therefore it must equal $\top$, which is true. This establishes $\mathtt{C} \vdash \mathsf{AlgNec}M \to \mathsf{WNec}M$. Since $\mathsf{AlgNec}M = \Box\mathsf{AlgNec}M$, we apply (CEquiv) to conclude $\mathtt{C} \vdash \mathsf{AlgNec} \leq \mathsf{Nec}$.

($\Leftarrow$) Assume $\mathsf{Nec}M$. Then clearly $M\top$, and because necessities are broadly necessarily necessities also $M\top = \top$. For distribution over conjunction, assume $M(p \wedge q)$. Now, $p \wedge q \to q$ is a tautology, so $M(p \wedge q \to q)$. By closure under modus ponens it follows that Mq. Likewise Mp using the tautology $p \wedge q \to p$. Conversely, assume $Mp \wedge Mq$. Since $p \to q \to (p \wedge q)$ is a tautology we have $M(p \to q \to (p \wedge q))$, so $M(q \to (p \wedge q))$ and in turn $M(p \wedge q)$ by closure under modus ponens. We have shown $\mathtt{C} \vdash \mathsf{Nec}M \to (M(p \wedge q) \leftrightarrow (Mp \wedge Mq))$, which by (CEquiv) and $\mathsf{AlgNec}M = \Box\mathsf{AlgNec}M$ yields $\mathtt{C} \vdash \mathsf{Nec}M \to (\lambda pq.M(p \wedge q)) = (\lambda pq.Mp \wedge Mq)$ as desired. Concluding, $\mathtt{C} \vdash \mathsf{Nec} \leq \mathsf{AlgNec}$. □

This result confirms that we are onto something. We have tried to formulate an object language notion of necessity from two different angles and ended up in the same place. There is a real logical joint here that deserves to be called the worldly analog of normal modal operators. Modal logics as deductive systems and modal algebras are two sides of the same coin, in the strong sense that every normal modal logic is sound and complete with respect to a class of modal algebras. Theorem 3.6 shows that this connection between proof theory and algebraic semantics internalizes. In view of Theorem 3.6, I will stop using the terms 'algebraic necessity' and 'necessity in Bacon's sense' and speak simply of necessities.

To further expand the algebraic perspective on necessities, I conclude this section by discussing two more characterizations of necessities, also of algebraic flavor. In Boolean algebras, there is a well known correspondence between *filters* and *congruences*. A *filter* on a Boolean algebra is a non-empty collection of elements that is closed under meets and upward closed with respect to the underlying order. A *congruence* on a Boolean algebra is an equivalence relation that is compatible with the Boolean operations: if it relates a to b, it must also relate the complement of a with the complement of b; if it relates a with a' and b with b', it must also relate the meet of a and b with the meet of a' and b', and so on. Filters induce congruence relations. A filter F determines a congruence defined by setting $a \sim_F b$ iff $a \leftrightarrow b$ belongs to F. Conversely, a congruence $\sim$ determines a filter $F_\sim$ defined as the collection of elements a such that $a \sim \top$. The correspondence is one-to-one: for any filter F, we have $F_{\sim_F} = F$, and for any congruence $\sim$, we have $\sim_{F_\sim} = \sim$.

Modal operators on Boolean algebras correspond closely to filters and congruences. In a modal algebra with modal operator $\blacksquare$, the relation that a bears to b when $\blacksquare(a \leftrightarrow b) = 1$ is a congruence, where 1 is the top element of the algebra. The collection of elements a such that $\blacksquare(a \leftrightarrow \top) = 1$ is a filter. Moreover, this filter and congruence are related by the above correspondence. Conversely, any filter F a Boolean algebra determines a modal operator $\blacksquare_F$ on the same, putting $\blacksquare_F a = 1$ iff a belongs to F, and 0 otherwise. Any congruence $\sim$ also determines a modal operator $\blacksquare_\sim$ on the same, putting $\blacksquare_\sim a = 1$ iff $a \sim \top$, and 0 otherwise. The modal operators determined by a filter and its corresponding congruence are the same.

The two characterizations of necessities I am about to give internalize these correspondences. The situation is somewhat different than in the case of modal algebras, again due to us working in an intensional framework. One feature of the correspondence outlined above is that only modal operators that range over exactly 0 and 1 are determined by filters and congruences. In the internalized setting, this is not true. There is, in fact, a one-to-one correspondence between necessities and what I shall call *modalized filters* and *modalized congruences*. Necessities literally are modalized filters. Any necessity determines the relation of being necessary equivalent in that sense of necessity, which is a modalized congruence. And any modalized congruence determines a necessity applying, as a matter of broad necessity, to precisely those proposition that it relates to the tautologous proposition.

Let us make this all precise.

Definition 3.7 (Modalized Filter)**.** A *modalized filter* is a property of propositions M that is necessarily non-empty, such that the relation two propositions bear when they are both M entails the relation they bear when their conjunction is M, and M itself equals the property of being entailed by some M:

$$\begin{aligned}
\mathsf{NE} &:= \lambda X.\exists p Xp = \top;\\
\mathsf{CC} &:= \lambda X.(\lambda pq.Xp \wedge Xq) \leq (\lambda pq.X(p \wedge q));\\
\mathsf{CE} &:= \lambda X.X = (\lambda p.\exists q.Xq \wedge q \leq p);\\
\mathsf{Filter!} &:= \lambda X.\mathsf{NE}X \wedge \mathsf{CC}X \wedge \mathsf{CE}X.
\end{aligned}$$

The first, second, and third conjunct internalize and modalize, respectively, the conditions that a filter be non-empty, closed under meets, and upward closed. Note that the second conjunct corresponds to the right-to-left direction of the second conjunct in the definition of an algebraic necessity. Each conjunct is equivalent to the claim that the corresponding extensional condition on filters, formalized in the obvious way, holds as a matter of broad necessity. The extension of a modalized filter necessarily satisfies the algebraic structure of a filter in the extensional sense. However, which propositions belong to the extension of a modalized filter is not fixed across modal space.

Our first characterization result relates necessities to modalized filters.

Proposition 3.8. *In Classicism, to be a necessity just is to be a modalized filter:*

$$\mathsf{C} \vdash \mathsf{Nec} = \mathsf{Filter!}.$$

Proof. ($\Rightarrow$) Assume $\mathsf{Nec}M$. The first two conjuncts of $\mathsf{Filter!}M$ obviously hold. For the third, obviously $Mp \to \exists q(Mq \wedge q \leq p)$. Conversely, if $\exists q(Mq \wedge q \leq p)$, then $p \wedge q = q$, so $M(p \wedge q)$. By distribution over conjunction, Mp. Conclude using (CEquiv) as before.

($\leftarrow$) Assume $\mathsf{Filter!}M$. By $\exists pMp = \top$, the logical truth $\forall p(p \leq \top)$ and closure under entailment we have $M\top = \top$. We only need the converse of the second conjunct of $\mathsf{Filter!}M$, which follows because $(p \wedge q) \to q$ and $(p \wedge q) \to p$ are tautologies and modalized filters are closed under entailment. □

This simple result will prove to be a very useful tool in the next section, where I study the algebraic structure of the family of necessities itself. Most importantly, I use it to establish a filter-theoretic characterization of the join of two necessities, which plays a crucial role in my proof that the modalized lattice of necessities is distributive.

Let me now turn to modalized congruences.

Definition 3.9 (Modalized Congruence)**.** A *modalized congruence* is a binary relation R on propositions that is intensionally reflexive and intensionally transitive, and that is compatible with conjunction and negation in the sense that (i) R is the same relation as the relation two propositions bear when their negations bear R, and (ii) the relation that four propositions bear when the first two and the second two are related by R entails the relation they bear when the conjunction of the first and the third bears R to the conjunction of the second and the fourth:

$$\begin{aligned}\mathsf{Congr!} := \lambda R.\mathsf{Refl!}R \wedge \mathsf{Trans!}R \wedge (R = (\lambda pq.R\neg p \neg q)) \wedge \\ (\lambda pqrs.Rpr \wedge Rqs) \leq (\lambda pqrs.R(p \wedge q)(r \wedge s)).\end{aligned}$$

The first and second conjuncts should be clear. The third conjunct internalizes and modalizes the claim that a congruence must the complements of equivalent elements equivalent. Likewise, the fourth conjunct does the same for the claim that a congruence must make the meets of pairwise equivalent pairs of things equivalent. Once more, each conjunct is equivalent to the claim that the corresponding extensional condition on congruences holds as a matter of broad necessity. Like modalized filters, modalized congruences may vary in extension across modal space.

Proposition 3.10. *In Classicism, there is a one-to-one correspondence between necessities and modalized congruences:*

$$\begin{aligned}
&\mathsf{C} \vdash \mathsf{Nec} \leq \lambda M.\mathsf{Congr!}(\lambda pq.M(p \leftrightarrow q)),\\
&\mathsf{C} \vdash \mathsf{Congr!} \leq \lambda R.\mathsf{Nec}(\lambda p.Rp\top),\\
&\mathsf{C} \vdash \mathsf{Nec} \leq (\lambda M.M = (\lambda p.M(p \leftrightarrow \top)))\\
&\mathsf{C} \vdash \mathsf{Congr!} \leq (\lambda R.R = (\lambda pq.R(p \leftrightarrow q)\top))
\end{aligned}$$

Proof. For the first item, assume $\mathsf{Nec}M$. We need to show that $\lambda pq.M(p \leftrightarrow q)$ is a modalized congruence. Modalized reflexivity and transitivity follow from the broadly necessary reflexivity and transitivity of material equivalence. That contraposition is a logical truth shows, via (CEquiv) that $\lambda pq.M(p \leftrightarrow q) = \lambda pq.M(\neg p \leftrightarrow \neg q)$. The last conjunct follows using the tautology $(p \leftrightarrow r) \wedge (q \leftrightarrow s) \to ((p \wedge q) \leftrightarrow (r \wedge s))$ using (CEquiv) again. Use (Equiv) to conclude the desired entailment.

For the second item, assume $\mathsf{Congr!}R$. We need to show that being R-equivalent to $\top$ is a necessity. By reflexivity, $R\top\top$. Moreover, $(p \wedge q \leftrightarrow \top) \leftrightarrow ((p \leftrightarrow \top) \wedge (q \leftrightarrow \top))$ is a tautology, as is $\top \wedge \top = \top$. The conclusion $\mathsf{Nec}M$ can be derived from these claims using (CEquiv). Use (Equiv) to conclude the desired entailment.

For the third item, assume $\mathsf{Nec}M$. We want to show that M equals the property of being M-necessarily equivalent to $\top$. Observe that Mp and Mq entails $M(p \wedge q)$, which in turn entails $M(p \leftrightarrow q)$ because $p \wedge q \leq p \leftrightarrow q$. In particular, Mp entails $M(p \leftrightarrow \top)$. Conversely, $M(p \leftrightarrow \top)$ entails Mp by closure under modus ponens. Use (CEquiv) followed by (Equiv) to conclude.

For the last item, assume $\mathsf{Congr!}R$. We want to show that the congruence corresponding to the necessity of being R-equivalent to $\top$ is R itself. It follows from the definition of a modalized congruence that

$$(\lambda pqrs.Rpr \wedge Rqs) \leq \lambda pqrs.R(p \leftrightarrow q)(r \leftrightarrow s).$$

So if Rpq, since also Rpp we have $R(p \leftrightarrow q)(p \leftrightarrow p)$. This is to say $R(p \leftrightarrow q)\top$. Conversely, assume $R(p \leftrightarrow q)\top$. By Rpp, it follows that $R(p \wedge p \leftrightarrow q)(p \wedge \top)$. Since $(p \wedge p \leftrightarrow q) \leftrightarrow (p \wedge q)$ and $p \wedge \top \leftrightarrow p$ are both tautologies this shows $R(p \wedge q)p$. By analogous reasoning, $Rq(p \wedge q)$, so by transitivity Rpq, as desired. Conclude using (CEquiv) and (Equiv). $\square$

Classicism proves Intensionalism, the principle that, as a matter of broad necessity, any two broadly necessarily equivalent propositions are identical:

$$\Box\forall pq(\Box(p \leftrightarrow q) \to p = q). \tag{Intensionalism}$$

We can see Intensionalism as a special case of the last item of Proposition 3.10. Identity is a modalized congruence. By definition, the necessity corresponding to identity is broad necessity, and the modalized congruence determined by the broadest necessity is broadly necessary equivalence. The last item of Proposition 3.10 tells us that the latter must be identity itself: when you transform a modalized congruence into a necessity and then back into a modalized congruence, you get the modalized congruence you started with.

4. The Intensional Lattice of Necessities

I now apply my algebraic characterization of necessities to study the (intensional) algebraic structure of the family of necessities itself. I review results due to Bacon and Zeng (2022) to the effect that necessities, like propositions, form an intensional bounded lattice with respect to entailment. I then apply my characterization of necessities as modalized filters to give a new characterization of arbitrary joins of persistent properties of necessities. This characterization yields a straightforward proof that the intensional lattice of necessities is *intensionally distributive*, in a sense to be specified. What is more, necessities satisfy an *infinitary* intensional distributivity principle that gives them the structure of what I call an *intensional Heyting algebra*.

4.1. **Bacon and Zeng's results.** Recall from Example 3.3 that the intensional and extensional meet and join of a rigid property of propositions X coincide. This generalizes to any relational type. In particular, extensional and intensional meets for rigid properties of type $(t \to t) \to t$. This implies that *binary* extensional meets and joins always coincide with their intensional counterparts, since the property of being identical to either of two entities is always rigid. Accordingly, I will henceforth speak simply of meets and joins in finitary contexts.

Bacon and Zeng (2022) proved that, in Classicism, the necessities form an intensional bounded lattice with respect to entailment.[2] Since the conjunction of two necessities is always a necessity, the meet of two necessities always equals their conjunction:

$$\mathtt{C} \vdash \Box \forall MN(\mathsf{Meet}(M \wedge_{t\to t} N)[M, N]\mathsf{Nec})$$

We thus let

$$\sqcap := \lambda XY.X \wedge_{t\to t} Y.$$

However, the join of two necessities is in general weaker than their disjunction, for the disjunction of two necessities is not always a necessity. For example, it might happen that p is X-necessary and $p \to q$ is Y-necessary. Since necessities are closed under modus ponens, the join of X and Y should make q necessary, yet q might lack both X and Y. Bacon and Zeng (2022) observed that the join of two necessities X and Y can be characterized as the property of having every necessity entailed by the disjunction of X and Y:

$$\sqcup := \lambda XYp.\forall Z(\mathsf{Nec}Z \wedge (X \vee_{t\to t} Y) \leq Z \to Zp).$$

We then have

$$\mathtt{C} \vdash \Box \forall MN(\mathsf{Join}(M \sqcup N)[M, N]\mathsf{Nec}).$$

So, Nec is a higher-order lattice. In addition, it is *bounded*: as a matter of broad necessity, there are a *broadest* and a *narrowest* necessity. As we already remarked, the broadest necessity is $\Box$. On the other hand, the narrowest necessity is

$$\mathsf{Deg} := (\lambda p.\top).$$

The narrowest necessity Deg makes every proposition necessary (and no proposition possible).

Finally, the intensional lattice of necessities satisfies fairly strong completeness principles. Bacon and Zeng established that every rigid property of necessities has an extensional meet, and every *persistent* property that necessarily only applies to necessities has an intensional meet:

$$\mathtt{C} \vdash \Box \forall X(\mathsf{Rigid}X \wedge X \leq \mathsf{Nec} \to \exists M(\mathsf{Meet}MX\mathsf{Nec}));$$
$$\mathtt{C} \vdash \Box \forall X(\mathsf{Persistent}X \wedge X \leq \mathsf{Nec}) \to \exists M(\mathsf{Meet!}MX\mathsf{Nec}).$$

In either case, the meet of the relevant property of necessity X coincides with the property of *having every necessity in X*:

$$\bigsqcap := \lambda Xp.\forall M(XM \to Mp).$$

[2]Bacon and Zeng's results do not mention the notion of an intensional lattice and are couched in a grain-neutral framework. But the results I attribute to them here are simple corollaries of their results as they stated them.

What about joins? The claim that every *persistent* property of necessities has an *intensional* join is an easy corollary of Bacon and Zeng's result about meets:

$$\mathsf{C} \vdash \Box\forall X(\mathsf{Persistent}X \wedge X \leq \mathsf{Nec} \to \exists M(\mathsf{Join!}MX\mathsf{Nec}));$$

When it exists, M coincides with the intensional meet of the property of being an intensional upper bound of X in Nec, which is itself persistent. The claim that every *rigid* property of necessities has an *extensional* join, as we shall see, is also true. But it is not an obvious corollary, since the property of being an extensional upper bound of X in Nec need not be rigid even if X is.

4.2. **Characterization of Intensional Joins.** I now apply Proposition 3.8 to characterize intensional joins of necessities. The idea is simple. In extensional contexts, joins of filters are easy to characterize: the join of two filters is just the filter generated by their union, or the set of all element in the underlying algebra that lie above the meet of two elements, one from each filter. We obtain a characterization of the join of two necessities simply by internalizing this construction.

Lemma 4.1. *In Classicism, the join of two necessities X and Y is the property of being entailed by the conjunction of some X-necessary proposition with some Y-necessary proposition:*

$$\mathsf{C} \vdash \Box\forall XY(\mathsf{Nec}[X,Y] \to X \sqcup Y = \lambda p.\exists qr(Xq \wedge Yr \wedge q \wedge r \leq p)).$$

Proof. Let X and Y be necessities and let Z be the property of being entailed by the conjunction of some X-necessity and some Y-necessity as defined on the right-hand side above. We first show that Z is a necessity. Since $\top$ is entailed by everything, Z applies to something. If $p \leq q$ and Zp, there must be q, r with Xq and Yr such that $q \wedge r \leq p$. But then also $q \wedge r \leq q$, whence Xq. Last, if Zp and Zq, there must be p', p'', q', q'' with Xp', Yp'', Xq', and Yq'' such that $p' \wedge p'' \leq p$ and $q' \wedge q'' \leq q$. Since both X and Y are closed under conjunction, we have $X(p' \wedge q')$ and $Y(p'' \wedge q'')$. Note also that $(p' \wedge p'') \wedge (q' \wedge q'') \leq p \wedge q$, whence $Z(p \wedge q)$. Use (CEquiv) to conclude that Z is a modalized filter, hence a necessity.

Now, since $p \wedge \top \leq p$ and $Y\top$ are broadly necessary, Xp entails Zp. Likewise, Yp entails Zp. This shows $X \sqcup Y \leq Z$ via (CEquiv). Conversely, assume Zp, so that there are q, r with Xq and Yr such that $q \wedge r \leq p$. Since both X and Y entail $X \sqcup Y$ and necessities are closed under conjunction, we have $X \sqcup Y(q \wedge r)$, whence $X \sqcup Yp$ by closure under entailment. This shows $Z \leq X \sqcup Y$ via (CEquiv). Conclude by generalizing and necessitating. $\Box$

This characterization can be generalized to arbitrary intensional joins. The rough idea is that an arbitrary intensional join $\bigsqcup X$ of a persistent property that broadly necessarily only applies to necessities is the property of being entailed by some finite conjunction $p_0 \wedge p_1 \wedge \ldots \wedge p_n$ such that each p_i is necessary according to some necessity falling under X. To make that precise, we need to regiment the idea of being a conjunction of finitely many propositions.

To do that, let us take a detour through the notion of a *numerical* quantifier (Bacon, Forthcoming). A numerical quantifier, intuitively, is a quantifier of the form 'there are exactly n Xs', for some n. In higher-order logic, they can be recursively defined thus as follows

$$\exists_0 := \lambda X.\neg\exists y Xy$$
$$\mathsf{Suc}Q := \lambda X.\exists x(Xx \wedge Q(\lambda y.Xy \wedge y \neq x)).$$

Using a standard Fregean trick, we then define the property of *being a numerical quantifier* as the property of having every property possessed by $\mathsf{E!}_0$ and closed under Suc:

$$\mathsf{NumQ} := \lambda Q.\forall X(X\exists_0 \wedge \forall Q'(XQ' \rightarrow X(\mathsf{Suc}Q'))).$$

We then let a *finite property* be one to which some numerical quantifier applies:

$$\mathsf{Fin} := \lambda X.\exists Q(\mathsf{NumQ}Q).$$

A finite conjunction is nothing but the meet of a finite property of propositions. We know that, in Classicism, the (intensional and extensional) meet of a *rigid* property of propositions X is just $\forall p.Xp \rightarrow p$ (and its least upper bound is $\exists p.Xp \wedge p$.) So, if we could show that any finite property of propositions is coextensive with a rigid property, then we would have our notion of a finite conjunction.

Predictably, this can be done. Say that a property X *rigidifies* a property Y when X is rigid and coextensive with Y:

$$\mathsf{Rigidifies} := \lambda XY.\mathsf{Rigid}X \wedge \forall z(Xz \leftrightarrow Yz).$$

We can show that every finite property is rigidified by something by showing that the property of *only applying to properties that have a rigidifier* is possessed by $\exists_0$ and closed under Suc, and so must be shared by all numerical quantifiers.

Proposition 4.2. *In Classicism, every finite property of propositions has a rigidifier:*

$$\mathsf{C} \vdash \Box\forall X(\mathsf{Fin}X^{t\rightarrow t} \rightarrow \exists Y(\mathsf{Rigidifies}YX)).$$

Proof. Consider the property:

$$F := \lambda Q.\forall X^{t\rightarrow t}(QX \rightarrow \exists Y(\mathsf{Rigidifies}YX)).$$

Note $F\exists_0$, because $\lambda p.\bot$ is rigid and coextensive with any property with empty extension. Assume FQ. If $\mathsf{Suc}QX$, that means

$$\exists p(Xp \wedge Q(\lambda q.Xq \wedge q \neq p)).$$

Fix one such p. By the inductive hypothesis, $\lambda q.Xq \wedge q \neq p$ has a rigidifier, say Y. But then $\lambda r.Yr \vee r = p$ is rigid and coextensive with X. This shows $F(\mathsf{Suc}Q)$, completing the induction. $\square$

In view of Proposition 4.2, quantification over finite properties of propositions can, without loss of generality, be replaced with quantification over *rigid* finite properties of propositions. So, by the *conjunction*, or meet, of a finite property of propositions X we will mean the proposition $\forall p(Yp \rightarrow p)$, where Y is a rigidifier of X. Likewise, by the *disjunction*, or join, of a finite property of propositions X we will mean the proposition $\exists p(Yp \wedge p)$, where Y is a rigidifier of X. These propositions always give the extensional and intensional meets and joins of any rigid property of propositions.

Using a similar inductive proof, we can show that all necessities are closed under finite conjunctions.

Proposition 4.3. *In Classicism, every necessity applies to the greatest lower bound of any finite, rigid property of proposition that only applies to propositions that are necessary in that sense of necessity:*

$$\mathsf{C} \vdash \mathsf{Nec}M \to \forall X(\mathsf{Finite}X \wedge \mathsf{Rigid}X \wedge X \subseteq M \to M(\forall p.Xp \to p)).$$

Proof. Again, we show this by establishing that only applying to properties that witness the consequent of the conditional above is a property possessed by $\exists_0$ and closed under Suc. If $\exists_0 X$, then $\forall p(Xp \to p) = \top$, so the property holds. Assume the property holds of Q. If $\mathsf{Suc}QX$, that means

$$\exists p(Xp \wedge Q(\lambda q.Xq \wedge q \neq p)).$$

Fix one such p. Then Mp. Let Y rigidify $\lambda q.Xq \wedge q \neq p$. Then Y is rigid, finite, and contained in X, whence $M(\forall q(Yq \to q))$ by the induction hypothesis. Since M is closed under conjunction, also $M(p \wedge \forall q(Yq \to q))$. But X, a rigid property, is coextensive, hence identical with $\lambda r.r = p \vee Yr$, so by closure under modus ponens we have $M(\forall q(Xq \to q))$, as desired. □

We need one more definition. When $X : (t \to t) \to t$ and $Y : t \to t$, we say that X *covers* Y when every Y proposition possesses one of the X-properties:

$$\mathsf{Covers} := \lambda X^{(t\to t)\to t} Y^{t \to t}.\forall p(Yp \to \exists Z(XZ \wedge Zp)).$$

When, in particular, X is a property of necessities, to say that X covers Y means that every Y-proposition is necessary according to some necessity possessing X.

We are now ready to generalize Lemma 4.1 to arbitrary joins.

Proposition 4.4. *In Classicism, the join of a persistent property X that broadly necessarily only applies to necessities is the property of being entailed by the conjunction of some rigid finite property of propositions that X covers:*

$$\mathsf{C} \vdash \Box\forall X(\mathsf{Persistent}X \wedge X \leq \mathsf{Nec} \to \bigsqcup X = \lambda p.\exists Z(\mathsf{Fin}Z \wedge \mathsf{Rigid}Z \wedge \mathsf{Covers}XZ \wedge \forall q(Zq \to q) \leq p).)$$

Proof. Let W be the property on the right-hand side of the identity above. Let X be as in the antecedent. If Mp for some M with XM, then $\lambda q.p = q \vee p = \top$ is a finite, rigid property covered by X whose greatest lower bound, namely p, entails p. That shows Wp, so necessitating and distributing yields $\bigsqcup X \leq W$. Conversely, take any necessity N with $M \leq N$ whenever XM. Assume Wp. Then there is a finite, rigid property Z covered by X such that $\forall q(Zq \to q) \leq p$. Because X covers Z, we have $Z \subseteq N$. So by Proposition 4.3, we must have $N(\forall q(Zq \to q))$. But then also Np by closure under entailment. This shows $W \leq \bigsqcup X$, and concludes the proof. □

4.3. **Distributivity and Heyting Algebra Structure.** A lattice is called *distributive* when meets and joins distribute over one another, the same way conjunction and disjunction do. In our intensional setting, the right notion to work with is that of an *intensionally distributive property* (with respect to entailment). It can be straightforwardly formulated as follows:

$$\mathsf{Distr!} := \lambda X.\Box\forall MNO(X[M,N,O] \to M \sqcap (N \sqcup O) = (M \sqcap N) \sqcup (M \sqcap O)).$$

There is a natural infinitary strengthening of this property, which we can call *complete* intensional distributivity:

$$\mathsf{CDistr!} := \lambda X.\Box\forall MY(XM \wedge Y \leq X \wedge \mathsf{Persistent}Y \rightarrow M \sqcap \bigsqcup X = \bigsqcup\nolimits_{XY} M \sqcap Y),$$

where

$$\bigsqcup\nolimits_{XY} M \sqcap Y := \bigsqcup \lambda Z.\exists Y(XY \wedge Z = M \sqcap Y).$$

While not explicitly appealing to my notions of intensional distributivity, Bacon and Zeng (2022, p. 177) raised the question of whether necessities form an intensionally distributive lattice, but left it open. In this section, I apply Proposition 4.4 to answer this question in the affirmative. In fact, I establish the stronger result that the intensional lattice of necessities is completely intensionally distributive.[3] It follows that necessities have the structure of an *intensional Heyting algebra*, in a sense to be specified.

While intensional distributivity is clearly a consequence of complete intensional distributivity, it is instructive to go through the simpler proof of the former first.

Theorem 4.5. *In Classicism, to be a necessity is an intensionally distributive property:*

$$\mathsf{C} \vdash \mathsf{DistrNec}.$$

Proof. Let M, N, O be as required by the antecedent of $\mathsf{DistrNec}$. The left-to-right direction holds in any lattice. For the converse, assume $(M \sqcap (N \sqcup O))\,p$. Then Mp and $N \sqcup Op$. So, there must be q, r with Nq and Or such that $q \wedge r \leq p$. Since M, N, O are closed under entailment, we have $M(p \vee q)$, $M(p \vee r)$, $N(p \vee q)$ and $O(p \vee r)$. But Classicism proves the identity $(p \vee q) \wedge (p \vee r) = p \vee (q \wedge r)$, which entails p because both disjunct do. This shows both $(M \sqcup N)p$ and $(M \sqcup O)p$, as desired. Use (CEquiv) to derive the desired identity, generalize and necessitate to conclude. $\square$

In the proof just given, we derive the distributivity of joins over binary meets of necessities by appealing to the distributivity of disjunction over binary conjunctions of propositions. The general case goes through because there is nothing special about the binary case: disjunction distributes over any finite conjunction, not just binary ones. We just had to be a little careful in properly formalizing finite conjunctions.

Theorem 4.6. *In Classicism, to be a necessity is a completely intensionally distributive property:*

$$\mathsf{C} \vdash \mathsf{CDistrNec}.$$

Proof. The $(\Leftarrow)$ direction holds in any lattice. For $(\Rightarrow)$ let X and M be as required by the antecedent of $\mathsf{CDistrNec}$. Assume $(M \sqcap \bigsqcup X)\,p$. Then Mp and $\bigsqcup Xp$. So, there must be some finite, rigid Y covered by X such that $\forall q(Yq \rightarrow q) \leq p$. Now, consider the property of being a disjunction of p with some Y-proposition,

$$O' := \lambda q.\exists r(Yr \wedge (q = p \vee r)).$$

[3] It is well known that the lattice of filters on a Boolean algebra is completely distributive, see, e.g., (Johnstone, 1982, II.2.11). Proving that Nec is completely intensionally distributive amounts to generalizing this fact to an intensional setting.

It is clearly finite. Let O be a rigidifier of O'. Notice that whenever Or holds, we have both Mr (because Mp and $p \leq r$) and Nr for some N with XN (because Yr and X covers Y). So, Oq implies $(M \sqcap N)q$ for some N with XN. This means that O is covered by $\lambda N \exists N'(XN' \wedge N = M \sqcap N)$, which $\bigsqcup_{XN} M \sqcap N$ is the join of. So, we are done if we can show that $\forall q(Oq \to q) \leq p$. This is indeed the case. Note that $\forall q(Oq \to q) = p \vee \forall q(Yq \to q)$. For the left-to-right entailment, observe that Classicism proves $p \vee \forall q(Yq \to q) = \forall q(Yq \to p \vee q)$, and the right-hand side clearly entails $\forall q(Oq \to q)$. Conversely, assume $\forall q(Oq \to q)$ and suppose $\neg p$. take any q with Yq. We know $O(p \vee q)$, so $p \vee q$. That can only be because q is true. Necessitating this reasoning shows the left-to-right entailment. But now $p \vee \forall q(Yq \to q) \leq p$ clearly holds, since both p and $\forall q(Yq \to q)$ entail p. Conclude as in the finite case. □

A *Heyting algebra* is a bounded distributive lattice in which, for any two elements a and b, there is a *pseudo-complement* of a relative to b, a weakest element $a \to b$ such that the meet of a and $a \to b$ is below b (Esakia, 2019, Ch. 2.1). Heyting algebras are the algebraic semantics of intuitionistic propositional logic, in the same way that Boolean algebras are the algebraic semantics of classical propositional logic. Relative pseudocomplements are used to interpret conditionals: the very definition of relative pseudocomplements can be seen as an algebraization of the introduction and elimination rules for the conditional.

Like we did for other algebraic properties, it is natural to introduce the concept of an *intensional Heyting algebra*. I will not define this concept in full generality; I will only define what it would take for necessities to form an intensional Heyting algebra. An *intensional relative pseudocomplement* operation on necessities would be a binary operation on necessities satisfying:

$$\begin{aligned}\mathsf{RPseudo!} := \lambda R.\Box\forall MN(\mathsf{Nec}[M,N] \to \mathsf{Nec}RMN)\wedge \\ \Box\forall MNO(\mathsf{Nec}[M,N,O] \to M \sqcap O \leq N \leftrightarrow M \leq RMN).\end{aligned}$$

Necessities form an intensional Heyting algebra if an intensional relative pseudocomplement operation on necessities exists.

In extensional contexts, every bounded and completely distributive lattice is a Heyting algebra. This generalizes to our intensional notions, allowing us to prove:

Theorem 4.7. *In Classicism, necessities form an intensional Heyting algebra, where the intensional relative pseudocomplement operation is given by*

$$\sqsupset := \lambda MN.\bigsqcup \lambda O.M \sqcap O \leq N.$$

Proof. Let M be a necessity. We first prove that $\lambda p.\mathsf{LB}\neg pM$ is a necessity. Since $\bot$ is a lower bound of everything, $\mathsf{LB}\top M$ holds. If $\mathsf{LB}\neg pM$ and $p \leq q$, then $\neg q \leq \neg p$, so $\mathsf{LB}\neg qM$ holds. If $\mathsf{LB}\neg pM$ and $\mathsf{LB}\neg qM$, then $\neg(p \wedge q) = \neg p \vee \neg q$ must also be a lower bound of M. Necessitating and distributing, we have proved that $\lambda p.\mathsf{LB}\neg pM$ is a necessity.

Now, observe that $M \sqcap (\lambda p.\mathsf{LB}\neg pM) = \Box$. For if both Mp and $\mathsf{LB}\neg pM$ hold, then $\neg p \leq p$, which can only be true if $p = \top$. Moreover, take any necessity N with $M \sqcap N = \Box$. This shows $(\lambda p.\mathsf{LB}\neg pM) \leq \overline{M}$. Conversely, assume $\overline{M}p$ and take any q with Mq. Then both $M(p \vee q)$ and $\overline{M}(p \vee q)$. Since $M \sqcap \overline{M} = \Box$, this implies $p \vee q = \top$. But, in turn, this implies $\neg p \leq q$, because $\neg p$ is the strongest proposition whose meet with p is $\top$. □

The fact that $\lambda O.M \sqcap O \leq N$ is a persistent property of necessities is a crucial ingredient in the proof, else we would not be able to show that its intensional join exists.

5. Pseudocomplements and Defective Necessities

Once we have relative pseudo-complements, we can define the *absolute* pseudo-complement of a necessity M as its pseudo-complement relative to $\Box$, the bottom element of our intensional lattice:

$$\overline{M} := M \sqsupset \Box.$$

We can think of $\overline{M}$ as the (intuitionistic) *negation* of a necessity M, in the same way that we can define the negation of a proposition p as $p \to \bot$.

A necessity M is said to satisfy the *Leibniz Biconditionals* when to be M necessary is to be true at all M-possible worlds. The notions of an M-possible world and that of truth at a world can be precisely defined in Classicism, but we do not need to get into the details here. The complement of a necessity that does satisfy the Leibniz Biconditionals is the property of being true at all *M-impossible worlds*. This modality has been studied model-theoretically in, e.g., Humberstone (1983); Gargov et al. (1987). But Classicism does not guarantee that all necessities satisfy the Leibniz Biconditionals (see Bacon and Dorr, 2024, Sec. 2). The claim that they do carries substantive metaphysical commitments. The pseudocomplement operation generalizes the same idea to all necessities, regardless of whether they satisfy the Leibniz Biconditionals. From truth at all M-impossible worlds, we generalize to *broadly necessarily being entailed by all M-impossible propositions*.

Proposition 5.1. *In Classicism, the pseudocomplement of a necessity M is the property of broadly necessarily being entailed by every M-impossible proposition—that is, of being an upper bound of the property of being M-impossible:*

$$\mathsf{C} \vdash \Box\forall M(\mathsf{Nec}M \to \overline{M} = \lambda p.\mathsf{UB}p(\lambda q.M\neg q))$$

Pseudocomplements in Nec do not in general behave classically. But some do. Call a necessity M *Boolean* when it satisfies the law of excluded middle, $M \sqcup \overline{M} = \mathsf{Deg}$, and *regular* when it is identical to its double pseudocomplement, $\overline{\overline{M}} = M$. Every Boolean necessity is regular, but what we have established about Nec so far leaves open whether there are regular, non-Boolean necessities. As it turns out, Classicism is consistent with the existence of necessities that are neither regular nor Boolean, as well as with the existence of regular, but non-Boolean necessities.

In the remainder of the paper, I describe the conditions under which necessities are Boolean or regular. I will show that being Boolean and being regular correspond to two different senses in which a necessity can be said to be *defective.* For illustration, take the example of the necessity *having probability 1* from the introduction. As I observed earlier, there are at least two senses in which this necessity is defective.

First, it is not *principal*: for every proposition that has probability 1, we can find a strictly stronger proposition that also has probability 1; there is no maximally strong proposition with probability 1. Suppose I play a lottery with countably many tickets $t_0, t_1, t_2, \ldots$ that has exactly one winning ticket. The probability that no ticket is the winner must be 0. Moreover, it is natural to think that for each individual ticket t_i, the probability that t_i is not the winning ticket is 1. Now, let p

be any proposition with probability 1. Then there must be a ticket t_i such that p does not entail that t_i is not the winning ticket, since otherwise p would entail that no ticket is the winner, which has probability 0. Given one such t_i, the conjunction of p with the proposition that t_i is not the winning ticket is strictly stronger than p and has probability 1.

The second defect in the necessity *having probability 1* is that it is not *infinitely closed*: while finitary consequences of propositions with probability 1 also have probability 1, there are infinitary consequences of propositions with probability 1 that fail to have probability 1. In the same scenario as before, the conjunction of the propositions that t_i is not the winning ticket for each i is an infinitary consequence of the propositions that each t_i is not the winning ticket. But clearly, it has probability 0, not 1.

I regiment the notions of principal and infinitely closed necessities and characterize them in terms of the classical principles on pseudocomplements that they correspond to: being Boolean corresponds to being principal (excluded middle), while being regular corresponds to being infinitely closed (double negation elimination).

5.1. **Principal Necessities.** Call a necessity M *principal* when it is *generated* by a single proposition, in the sense that, broadly necessarily, there is a proposition p that entails precisely the M-necessary propositions:

$$\mathsf{Princ} := \lambda M.\Box\exists p(\forall q(Mq \leftrightarrow p \leq q)).$$

Equivalently, a necessity is principal when, broadly necessarily, it has an extensional meet and makes it necessary:

$$\mathtt{C} \vdash (\mathsf{Nec} \wedge \mathsf{Princ}) = \lambda M.\mathsf{Nec}M \wedge \Box\exists p(\mathsf{Meet}pM \wedge Mp).$$

Indeed, if M is principal, then its extensional meet is whichever proposition generates it. And conversely, any necessity that broadly necessarily makes its extensional meet necessary must be generated by the same.

Intuitively, then, a principal necessity is one that admits a single strongest necessary proposition, as a matter of broad necessity. This proposition can, but does not have to be the same across worlds, so to speak. For example, *being entailed by Alice's favorite proposition* is a principal necessity, even if Alice does not have the same favorite proposition across all worlds. Any property of the form

$$\Box_p := \lambda q.p \leq q,$$

for any proposition p. In particular, $\Box = \Box_\top$ and $\mathsf{Deg} = \Box_\bot$, so both are principal.

An easy way to construct concrete examples of non-principal necessities is to take a collection of propositions closed under conjunction and entailment that have a greatest lower bound, which however cannot be expressed as the *finite* conjunction of propositions in the collection. Here is one such example. I point the reader to Bacon (2023) for details about the semantics of higher-order logic.

Example 5.2. Construct a model of higher-order logic as follows. We think of natural numbers as worlds. Let the domain of propositions $\mathbf{A}^t$ consist of all sets of natural numbers, let the domain of individuals be arbitrary and let higher order domains $\mathbf{A}^\sigma \to \mathbf{A}^\tau$ consist of all functions from $\mathbf{A}^\sigma$ to $\mathbf{A}^\tau$. We interpret Boolean connectives as set-theoretic operations and quantifiers as infinite intersections and unions. Consider now the function $\mathbf{f}$ that maps a set of natural numbers to $\mathbb{N}$ iff it is cofinite and includes all even numbers, and to $\varnothing$ otherwise. In our model, this function is a necessity: the intersection of any two such sets is again cofinite and includes all even numbers, as is any superset thereof. But the intersection of all cofinite sets containing all even numbers is the set of even numbers itself, which is not cofinite. So, $\mathbf{f}$ is not principal: while its extensional meet exists, $\mathbf{f}$ does not make it necessary.

As our first application of Proposition 5.1, I show that the Boolean necessities are precisely the principal ones. We first apply Proposition 5.1 to characterize the pseudocomplement of a principal necessity, then use it to derive the desired identity.

Proposition 5.3. *In Classicism, the pseudo-complement of a principal necessity M is the property of being entailed by the negation of its greatest lower bound:*

$$\mathsf{C} \vdash \Box\forall M(\mathsf{Nec}M \wedge \mathsf{Princ}M \rightarrow \overline{M} = \lambda q.\exists p(\mathsf{Meet}pM \wedge \neg p \leq q)).$$

Proof. Let M be a principal necessity and p its meet. We know from Proposition 5.1 that $\overline{M}$ equals the property of being entailed by every M-impossible proposition. But if p is the meet of M—that is, the strongest M-necessary proposition—then $\neg p$ must be the weakest M-impossible proposition. So, to be entailed by $\neg p$ is to be entailed by every M-impossible proposition. $\square$

Theorem 5.4. *In Classicism, to be a Boolean necessity just is to be a principal necessity:*

$$\mathsf{C} \vdash (\mathsf{Nec} \wedge \mathsf{Boolean}) = (\mathsf{Nec} \wedge \mathsf{Princ}).$$

Proof. Assume M is a principal necessity. Let p be the greatest lower bound of M. By Proposition 5.1, $\overline{M}\neg p$, so because $p \wedge \neg p = \bot$, we have $(M \sqcup \overline{M})\bot$ by Lemma 4.1. Necessitating this reasoning shows $(M \sqcup \overline{M}) = \mathsf{Deg}$. Moreover, again using Proposition 5.1, note that $(M \sqcap \overline{M})q$ holds iff $p \leq q$ and $\neg p \leq q$, which can only be if $q = \top$. Necessitating yields $M \sqcap \overline{M} = \Box$. So, M is Boolean.

Conversely, let M be a Boolean necessity. Then $M \sqcup \overline{M} = \mathsf{Deg}$. By Lemma 4.1 and the fact that $\mathsf{Deg}\bot$, it follows that there must be some p, q such that Mp and $\overline{M}q$ and $p \wedge q = \bot$. This implies $q \leq \neg p$, so that $\overline{M}(\neg p)$. I claim that M is generated by p. Indeed, take any q with Mq. Then $M(\neg p \vee q)$ and $\overline{M}(\neg p \vee q)$ by closure under entailment. Since $M \sqcap \overline{M} = \Box$, that means $\neg p \vee q = \top$, which is to say $p \leq q$. Necessitating this reasoning yields the left-to-right entailment. $\square$

Proposition 5.3 draws a nice connection between pseudocomplementation and negation: the pseudocomplement of a Boolean necessity is the necessity generated by the negation of its greatest lower bound. This connection is not unique to pseudocomplementation; it extends to meets and joins as well, relating meets to disjunctions and joins to conjunctions. The *meet* of two Boolean necessities is just the necessity generated by the *disjunction* of their greatest lower bounds, and the *join* of two Boolean necessities is just the necessity generated by the *conjunction* of their greatest lower bounds. This shows, transparently, that Boolean necessities form a Boolean subalgebra of the algebra of necessities.

Proposition 5.5. *If M, N are Boolean necessities, then $M \sqcup N$ equals the property of being entailed by the disjunction of the greatest lower bounds of M and N, and*

$M \sqcap N$ equals the property of being entailed by the conjunction of the greatest lower bounds of M and N. Consequently, both are Boolean.

$$\mathtt{C} \vdash \Box\forall MN(\mathsf{Nec} \wedge \mathsf{Princ})[M,N] \rightarrow (M \sqcap N = (\lambda p.\exists qr(\mathsf{Meet}qM \wedge \mathsf{Meet}rN \wedge (q \vee r) \leq p)))$$

$$\mathtt{C} \vdash \Box\forall MN(\mathsf{Nec} \wedge \mathsf{Princ})[M,N] \rightarrow (M \sqcup N = (\lambda p.\exists qr(\mathsf{Meet}qM \wedge \mathsf{Meet}rN \wedge (q \wedge r) \leq p)))$$

Proof. Let M, N be Boolean necessities. For meets, observe that p is entailed by both the greatest lower bound of M and that of N iff it is entailed by their disjunction. For joins, if $(N \sqcup M)p$, by Lemma 4.1 there are q, r with Mq and Nr such that $q \wedge r \leq p$. The greatest lower bound of M entails q and the greatest lower bound of N entails r, so their conjunction entails p. The converse follows similarly from Lemma 4.1. □

By similar reasoning we can show that, in the general case, when X is any persistent property that entails being a Boolean necessity, its meet is principal iff the disjunction of all the greatest lower bounds of some necessity in X exists. When that happens, the join equals the property of being entailed by said disjuinction. For said disjunction is entailed by every lower bound of some necessity in M, and so must belong to the meet by closure under entailment. On the other hand, the join of X can fail to be Boolean even when the conjunction of all greatest lower bounds of necessities in X exists. For the join of X must be broadly necessarily the *weakest* necessity broadly necessarily entailed by every necessity in X, and the weakest such necessity will in general not be closed under arbitrary conjunctions.

5.2. **Regular Necessities.** Let us now turn to regular necessities, those equal to their double pseudocomplement. Obviously, for any necessity M, broadly necessarily, if Mp, then p is entailed by every lower bound of M. The converse need not be true. When it is, we call M a *cut*:[4]

$$\mathsf{Cut} := \lambda M.\Box\forall p(Mp \leftrightarrow \forall q(\mathsf{LB}qM \rightarrow q \leq p)).$$

Intuitively, a cut is a necessity that can be uniquely reconstructed from its lower bounds: anything entailed by all lower bounds is already necessary according to the cut.

Any principal necessity is a cut:

$$\mathtt{C} := (\mathsf{Nec} \wedge \mathsf{Princ}) \leq (\mathsf{Nec} \wedge \mathsf{Cut}).$$

For if M necessarily has a greatest lower bound p, then q is entailed by p iff it is entailed by every lower bound of M. However, intuitively a necessity can be a cut without being principal if it possibly lacks an extensional meet. Consider, for example, the Boolean algebra of finite and cofinite subsets of $\mathbb{N}$. Let E be the function mapping every cofinite subset of $\mathbb{N}$ that contains every even number to $\mathbb{N}$ (the top element of the algebra), and everything else to $\varnothing$. Then E is a necessity. It is also a cut: the lower bounds of E are precisely the finite subsets of even numbers, and a set contains (is entailed by) every finite subset of even numbers iff it contains the set of even numbers itself. The set of even numbers itself does not live in our algebra—only its cofinite supersets do, which are all mapped by E to

[4]The name 'cut' is inspired by the literature on completions of posets, where a set is called a cut if it equals the set of all upper bounds of all its lower bounds. These sets are also called *Dedekind-Macneille closed*. See, e.g., Davey and Priestley (2002, Ch. 7.38)

the top element $\mathbb{N}$. However, E is not principal. Because, as we just saw, its lower bounds are precisely the finite subsets of even numbers, E has no extensional meet, and so cannot be principal.

This argument is not a *proof* that, in the object language, not every cut is a principal necessity. To prove that, we would have to show that the model we relied on can be extended to a full model of Classicism that supports the same reasoning. This particular example does not seem to admit such an extension.[5] To actually prove that not every cut is principal, we can consider countermodels to the principle of *Boolean Completeness* at type $t \to t$, which says that every property of propositions has an extensional meet:

$$\forall X^{t\to t}\exists p \mathsf{Meet} pX. \qquad \text{(Boolean Comp)}$$

A model of Classicism that refutes (Boolean Comp) must contain a property F that lacks an extensional meet. One can then use F to construct a non-principal cut, obtained by closing F under the *multi-premise entailment* relation defined below. Through such an argument we thus establish:

Theorem 5.6. *In Classicism, not every cut is a principal necessity:*

$$\mathtt{C} \vdash \exists M(\mathsf{Nec}M \wedge \mathsf{Cut}M \wedge \neg\mathsf{Princ}M).$$

While it is known that Boolean Comp is independent of Classicism, countermodels to Boolean Comp are less than straightforward to construct. I offer one in **??** and explain how to construct a non-principal cut in detail.

To say that p is entailed by every lower bound of M is a natural regimentation of the idea that M, a potentially infinite property of proposition, *multi-premise entails* p. Indeed, Bacon (2023, Def. 7.7) defines multi-premise entailment exactly this way:

$$\trianglelefteq := \lambda X^{t\to t}p.\forall q(\mathsf{LB}qX \to q \leq p).$$

Every M-necessary proposition is infinitely closed in the sense that it makes necessary any multi-premise consequence of M-necessary propositions:

$$\mathtt{C} \vdash \Box\forall M(\mathsf{Nec}M \wedge \mathsf{Cut}M \to \Box\forall Xp(X \subseteq M \wedge X \trianglelefteq p \to Mp)).$$

What is more, it turns out that the regular necessities are precisely the cuts. If we compute the double pseudocomplement of a necessity M using Proposition 5.1, we get exactly the property of being entailed by every lower bound of M, which implies the desired identity.

Theorem 5.7. *In Classicism, the double pseudocomplement of a necessity M is the property of being entailed by every lower bound of M:*

$$\mathtt{C} \vdash \Box\forall M(\mathsf{Nec}M \to \overline{\overline{M}} = (\lambda p.\forall q(\mathsf{LB}qM \to q \leq p))).$$

Consequently, to be a regular necessity is just to be a cut:

$$\mathtt{C} \vdash (\mathsf{Nec} \wedge \mathsf{Regular}) = (\mathsf{Nec} \wedge \mathsf{Cut}).$$

[5]When constructing a model of classicism based on it that shows what we hope to show, we would have to put the function E in the domain at type $t \to t$. But if we do that, the set of even numbers would belong to the domain at type t, because it would equal the proposition that every E-proposition is true. Still, I suspect that some variant of the argument just sketched can be made to work.

Proof. Let M be a necessity. By Proposition 5.1, $\overline{\overline{\overline{M}}}$ equals the property of being the negation of a lower bound $\overline{M}$. But using Proposition 5.1 again, a lower bound of $\overline{M}$ is just a proposition that entails the negation of every lower bound of M, so its negation is a proposition entailed by every lower bound of M. □

Thus, the double pseudocomplementation operation coincides with the operation of closing a necessity under multi-premise entailment.

Regular necessities also form a Boolean algebra, though this need not be a subalgebra of Nec. The meet of two regular necessities is always regular, as double pseudo-complementation commutes with finite meets. The pseudocomplement of a regular necessity is also regular. However, the join of two regular necessities need not be regular. Intuitively, this is so because the lower bounds of $M \sqcup N$ are precisely those propositions that entail all finite conjunctions of M-necessities with N-necessities. But only the consequences of such finite conjunctions are necessary according to $M \sqcup N$.

To verify this, it is helpful to appeal to the fact that double-complementation is a *closure operator*: an increasing, monotonic and idempotent operator.

Proposition 5.8. *In Classicism,* $\bar{\bar{\cdot}}$ *is a closure operator on* Nec*:*

$$\mathtt{C} \vdash \mathsf{Nec}M \to M \leq \overline{\overline{M}}$$

$$\mathtt{C} \vdash \mathsf{Nec}M \wedge M \leq N \to \overline{\overline{M}} \leq \overline{\overline{N}}$$

$$\mathtt{C} \vdash \mathsf{Nec}M \to \overline{\overline{\overline{\overline{M}}}} = \overline{\overline{M}}.$$

Proof. Double pseudo-complementation is a closure operator in any Heyting algebra. In our setting, all identities above are also easily established by reasoning about lower bounds. Every M-necessity is entailed by every lower bound of M. If $M \leq N$, then every lower bound of N is a lower bound of M. Last, $\overline{\overline{\overline{\overline{M}}}}p$ holds iff p is entailed by all lower bounds of $\overline{\overline{M}}$, which are precisely the lower bounds of M. □

Proposition 5.9. *In Classicism, the regular necessities form a Boolean algebra. Meets and complements are given by* $\sqcap$ *and* $\bar{\cdot}$ *respectively, while the join of* M *and* N *is given by* $\overline{\overline{M \sqcup N}}$*:*

$$\mathtt{C} \vdash \Box\forall MN(\mathsf{Nec} \wedge \mathsf{Regular})[M,N] \to \mathsf{Regular}(M \sqcap N) \wedge \mathsf{Regular}(\overline{M})$$

$$\mathtt{C} \vdash \Box\forall MN(\mathsf{Nec} \wedge \mathsf{Regular})[M,N] \to \mathsf{Meet}\overline{\overline{M \sqcup N}}(\lambda O.O = M \vee O = N)(\mathsf{Nec} \wedge \mathsf{Regular})$$

$$\mathtt{C} \vdash \Box\forall M(\mathsf{Nec}M \wedge \mathsf{Regular}M \to \overline{\overline{M \sqcup \overline{M}}} = \mathsf{Deg}).$$

Proof. Let M, N be regular necessities. For meets, simply observe that the lower bounds of $M \sqcap N$ are exactly the propositions that are lower bounds of all propositions that are both M- and N-necessary, so anything entailed by these is already necessary according to both M and N. For joins, obviously $\overline{\overline{M \sqcup N}}$ is regular, and because $\bar{\bar{\cdot}}$ is increasing it is also a lower bound of M and N. That it is the least such follows from the monotonicity and idempotence of $\bar{\bar{\cdot}}$.

For complements, assume p is entailed by every lower bound of $\overline{M}$. If q is a lower bound of $\overline{M}$, then by Proposition 5.1 $\neg q$ is entailed by every lower bound of M, which is to say Mq because M is regular. So, to say that p is entailed by every lower bound of $\overline{M}$ is to say that $\neg p$ entails every M-necessary proposition, i.e., it is a lower bound of M. This shows $\overline{M}p$ by Proposition 5.1.

For the last item, assume p is a lower bound of $M \sqcup \overline{M}$. Thus, p entails every conjunction of an M-necessary proposition with an $\overline{M}$-necessary proposition. In particular, it is a lower bound of both M and $\overline{M}$. By Proposition 5.1 it follows that $\overline{M}\neg p$ and $\overline{\overline{M}}\neg p$. Because M is regular, it follows that $M\neg p$. But $M \sqcap \overline{M} = \Box$, so $\neg p = \top$ and so $p = \bot$. $\square$

References

Bacon, A. [2018]. The Broadest Necessity. *Journal of Philosophical Logic*, 47(5):733–783.

——— [2023]. *A Philosophical Introduction to Higher-order Logics*. Oxford University Press.

——— [Forthcoming]. Could the Truths of Mathematics Have Been Different? *Philosophical Review*.

Bacon, A. and Dorr, C. [2024]. Classicism. In P. Fritz and N. K. Jones, eds., *Higher-order Metaphysics*. Oxford University Press.

Bacon, A. and Zeng, J. [2022]. A Theory of Necessities. *Journal of Philosophical Logic*, 51(1):151–199.

Chagrov, A. and Zakharyaschev, M. [1997]. *Modal Logic*. New York: Clarendon Press.

Davey, B. A. and Priestley, H. A. [2002]. *Introduction to Lattices and Order*. Cambridge University Press, second ed.

Esakia, L. [2019]. *Heyting Algebras: Duality Theory*, *Trends in Logic*, vol. 5. Amsterdam: Springer.

Gargov, G., Passy, S., and Tinchev, T. [1987]. Modal Environment for Boolean Speculations. pp. 253–263. Boston, MA: Springer US.

Hale, B. [2013]. *Necessary Beings: An Essay on Ontology, Modality, and the Relations Between Them*. Oxford, England: Oxford University Press.

Hale, B. and Leech, J. [2017]. Relative Necessity Reformulated. *Journal of Philosophical Logic*, 46(1):1–26.

Humberstone, I. L. [1983]. Inaccessible worlds. *Notre Dame Journal of Formal Logic*, 24(3):346–352.

Johnstone, P. T. [1982]. *Stone Spaces*. Cambridge: Cambridge University Press.

Kripke, S. A. [1980]. *Naming and Necessity*. Harvard University Press.

Lewis, D. K. [1986]. *On the Plurality of Worlds*. Oxford: Blackwell.

Roberts, A. [2020]. Relative Necessity and Propositional Quantification. *Journal of Philosophical Logic*, 49(4):703–726.

——— [2025]. Necessity in the Highest Degree. *Journal of Philosophical Logic*, 54(1):51–97.

Williamson, T. [2016]. Modal Science. *Canadian Journal of Philosophy*, 46(4-5):453–492.